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Mityagin's extension problem. Progress report ☆

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ABSTRACT

Given a compact set $K \subset \mathbb{R}^d$, let $\mathcal{E}(K)$ denote the space of Whitney jets on K. The compact set K is said to have the extension property if there exists a continuous linear extension operator $W : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$. In 1961 B.S. Mityagin posed a problem to give a characterization of the extension property in geometric terms. We show that there is no such complete description in terms of densities of Hausdorff contents or related characteristics. Also the extension property cannot be characterized in terms of growth of Markov's factors for the set.

1. Introduction

By the celebrated Whitney theorem [24], for each compact set $K \subset \mathbb{R}^d$, by means of a continuous linear operator one can extend jets of finite order from $\mathcal{E}^p(K)$ to functions defined on the whole space, preserving the order of differentiability. In the case $p = \infty$, the possibility of such extension crucially depends on the geometry of the set. Following [21], let us say that K has the *extension property (EP)* if there exists a linear continuous extension operator $W : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$. Clearly, there always exists a linear extension operator (one can individually extend the elements of a vector basis in $\mathcal{E}(K)$) and a continuous extension operator, by Whitney's construction. Numerous examples show that a set K has EP provided "local thickness" of K. For example, any set K with an isolated point does not have EP ([14], Prop. 21).

B.S. Mityagin posed in 1961 ([14], p. 124) the following problem (in our terms):

What is a geometric characterization of the extension property?

We show that there is no complete characterization of that kind in terms of densities of Hausdorff contents of sets or analogous functions related to Hausdorff measures.

This is similar to the state in Potential Theory where R. Nevanlinna [15] and H. Ursell [22] proved that there is no complete characterization of polarity of compact sets in terms of Hausdorff measures. The scale of growth rate of functions h, which define the Hausdorff measure Λ_h, can be decomposed into three zones.

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For h from the first zone of small growth, if $0 < \Lambda_h(K)$ then the set K is not polar. For h from the zone of fast growth, if $\Lambda_h(K) < \infty$ then the set K is polar. However, there is a zone of uncertainty between them. It is possible to take two functions with $h_2 \prec h_1$ from this zone and the corresponding Cantor-type sets K_j with $0 < \Lambda_{h_j}(K_j) < \infty$ for $j \in \{1, 2\}$, such that the large (with respect to the Hausdorff measure) set K_2 is polar, whereas the smaller K_1 is not polar.

Here we present a similar example of two Cantor-type sets: the smaller set has EP whereas the larger set does not.

Of course, such global characteristics as Hausdorff measures or Hausdorff contents cannot be used, in general, to distinguish EP, which is defined by a local structure of the set. One can suggest for this reason to characterize EP in terms of lower densities of Hausdorff contents of sets, because (see Section 9) densities of Hausdorff measures cannot be used for this aim. We analyze a wide class of dimension functions and show that lower densities of Hausdorff contents do not distinguish EP.

Neither EP can be characterized in terms of growth rate of Markov's factors $(M_n(\cdot))_{n=1}^{\infty}$ for sets. Two sets are presented, K_1 with EP and K_2 without it, such that $M_n(K_1)$ grows essentially faster than $M_n(K_2)$ as $n \to \infty$. It should be noted that, by W. Pleśniak [17], any Markov compact set (with a polynomial growth rate of $M_n(\cdot)$) has EP. All examples are given in terms of the sets $K(\gamma)$ introduced in [10]. The paper sums up the research related to the problem by the first author in the last two decades.

The organization of the paper is the following. Section 2 is a short review of main methods of extension; in it we also consider the Tidten–Vogt linear topological characterization of EP. In Section 3 we give some auxiliary results about the weakly equilibrium Cantor-type set $K(\gamma)$. In Section 4 we use local Newton interpolations to construct an extension operator W. Section 5 contains the main result, namely a characterization of EP for $\mathcal{E}(K(\gamma))$ in terms of a sequence related to γ. In section 6 we compare W with the extension operator from [12], which is given by individual extensions of elements of Schauder basis for the space $\mathcal{E}(K(\gamma))$. In Section 7 we consider two examples that correspond respectively to regular and irregular behaviour of the sequence γ. In Section 8 we calculate $\Lambda_h(K(\gamma))$ for the dimension function h that corresponds to the set and show that $\Lambda_h|_{K(\gamma)}$ coincides with the equilibrium measure of $K(\gamma)$. Also in this section we present Ursell's type example for EP. In Section 9 we consider Hausdorff contents and related characteristics. In Section 10 we compare the growth of Markov's factors and EP for $K(\gamma)$.

For the basic facts about the spaces of Whitney functions defined on closed subsets of $\mathbb{R}^d$ see e.g. [3], the concepts of the theory of logarithmic potential can be found in [18]. Throughout the paper, log denotes the natural logarithm. Given compact set K, $Cap(K)$ stands for the logarithmic capacity of K, $Rob(K) = \log(1/Cap(K)) \le \infty$ is the Robin constant for K. If K is not polar then μ_K is its equilibrium measure. For each $A \subset \mathbb{R}$, let $\#(A)$ be the cardinality of A, $|A|$ be the diameter of A. Given a finite set $A = (a_m)$ and $x \in \mathbb{R}$, by $(d_k(x, A))$ we denote distances from x to the points of A arranged in nondecreasing order, so $d_k(x, A) = |x - a_{m_k}| \nearrow$. Also, $[a]$ is the greatest integer in a, $\sum_{k=m}^{n}(\cdots) = 0$ and $\prod_{k=m}^{n}(\cdots) = 1$ if $m > n$. The symbol $\sim$ denotes the strong equivalence: $a_n \sim b_n$ means that $a_n = b_n(1 + o(1))$ for $n \to \infty$.

2. Three methods of extension

Let $K \subset \mathbb{R}^d$ be a compact set, $\alpha = (\alpha_j)_{j=1}^{d} \in \mathbb{N}_0^d$ be a multi-index. Let I be a closed cube containing K and $\mathcal{F}(K, I) = \{F \in C^\infty(I) : F^{(\alpha)}|_K = 0, \forall \alpha\}$ be the ideal of flat on K functions. The Whitney space $\mathcal{E}(K)$ of extendable jets consists of traces on K of C^∞-functions defined on I, so it is a factor space of $C^\infty(I)$ and the restriction operator $R : C^\infty(I) \longrightarrow \mathcal{E}(K)$ is surjective. This means that the sequence $0 \longrightarrow \mathcal{F}(K, I) \xrightarrow{J} C^\infty(I) \xrightarrow{R} \mathcal{E}(K) \longrightarrow 0$ is exact. If it splits, then the right inverse to R is the desired linear continuous extension operator W and K has EP.

In [21] M. Tidten applied D. Vogt's theory of splitting of short exact sequences of Fréchet spaces (see e.g. [13], Chapter 30) and presented the following important linear topological characterization of EP:

a compact set K has the extension property if and only if the space $\mathcal{E}(K)$ has a dominating norm (satisfies the condition (DN)).

Recall that a Fréchet space X with an increasing system of seminorms $(\,||\cdot||_k)_{k=0}^{\infty}$ has a *dominating norm* $||\cdot||_p$ if for each $q \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and $C \geq 1$ such that $||\cdot||_q^2 \leq C\,||\cdot||_p\,||\cdot||_r$.

Concerning the question "How to construct an operator W if it exists?", we can select three main methods that can be applied for wide families of compact sets.

The first method goes back to B.S. Mityagin [14]: to extend individually the elements $(e_n)_{n=1}^{\infty}$ of a topological basis of $\mathcal{E}(K)$. Then for $f = \sum_{n=1}^{\infty} \xi_n \cdot e_n$ take $W(f) = \sum_{n=1}^{\infty} \xi_n \cdot W(e_n)$. See Theorem 2.4 in [23] about possibility of suitable simultaneous extensions of e_n in the case when K has a nonempty interior. The main problem with this method is that we do not know whether each space $\mathcal{E}(K)$ has a topological basis, even though $\mathcal{E}(K)$ is complemented in $C^{\infty}(I)$. This is a particular case of the significant Mityagin–Pełczyński problem: Suppose X is a nuclear Fréchet space with basis and E is a complemented subspace of X. Does E possess a basis? The space $X = s$ of rapidly decreasing sequences, which is isomorphic to $C^{\infty}(I)$, presents the most important unsolved case.

The second method was suggested in [16], where W. Pawłucki and W. Pleśniak constructed an extension operator W in the form of a telescoping series containing Lagrange interpolation polynomials with Fekete nodes. The authors considered the family of compact sets with polynomial cusps, but later, in [17], the result was generalized to any Markov set. In fact (see T.3.3 in [17]), for each C^{∞} determining compact set K, the operator W is continuous in the so-called Jackson topology τ_J if and only if τ_J coincides with the natural topology τ of the space $\mathcal{E}(K)$ and this happens if and only if the set K is Markov. We remark that τ_J is not stronger than τ and that τ_J always has the dominating norm property, see e.g. [2]. Thus, in the case of non-Markov compact set with EP ([5,2]), the Pawłucki–Pleśniak extension operator is not continuous in τ_J, yet this does not exclude the possibility for it to be bounded in τ. At least for some non-Markov compact sets, the local version of this operator is bounded in τ ([2]).

In [4] L. Frerick, E. Jordá, and J. Wengenroth showed that, provided some conditions, the classical Whitney extension operator for the space of jets of finite order can be generalized to the case $\mathcal{E}(K)$. Instead of Taylor's polynomials in the Whitney construction, the authors used a kind of interpolation by means of certain local measures. A linear tame extension operator was presented for $\mathcal{E}(K)$, provided K satisfies a local form of Markov's inequality.

There are some other methods to construct W for closed sets, for example Seeley's extension [19] from a half space or Stein's extension [20] from sets with the Lipschitz boundary. However these methods, in order to define $W(f,x)$ at some point x, essentially require existence of a line through x with a ray where f is defined, so these methods cannot be applied for compact sets.

Here we consider rather small Cantor-type sets that are neither Markov nor local Markov. We follow [2] in our construction, so W is a local version of the Pawłucki–Pleśniak operator. It is interesting that, at least for small sets, W can be considered also as an operator extending basis elements of the space. Thus, for such sets, the first method and a local version of the second method coincide.

3. Notations and auxiliary results

In what follows we will consider only perfect compact sets $K \subset I = [0,1]$, so the Fréchet topology τ in the space $\mathcal{E}(K)$ can be given by the norms

$$||\, f\,||_q = |f|_{q,K} + sup\left\{ \frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-k}} : x, y \in K, x \neq y, k = 0, 1, \ldots, q \right\}$$

for $q \in \mathbb{N}_0$, where $|f|_{q,K} = sup\{|f^{(k)}(x)| : x \in K, k \leq q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder.

Given $f \in \mathcal{E}(K)$, let $|||\, f \,|||_q = inf \;|\, F \,|_{q,I}$, where the infimum is taken over all possible extensions of f to $F \in C^\infty(I)$. By the Lagrange form of the Taylor remainder, we have $||\, f \,||_q \le 3 \,|\, F \,|_{q,I}$ for any extension F. The quotient topology τ_Q, given by the norms $(|||\cdot|||_{q=0}^\infty)$, is complete and, by the open mapping theorem, is equivalent to τ. Hence, for any q there exist $r \in \mathbb{N}$, $C > 0$ such that

$$|||\, f \,|||_q \le C \,||\, f \,||_r \tag{1}$$

for any $f \in \mathcal{E}(K)$. In general, extensions F that realize $|||\, f \,|||_q$ for a given function f, essentially depend on q. Of course, the extension property of K means the existence of a simultaneous extension which is suitable for all norms.

Our main subject is the set $K(\gamma)$ introduced in [10]. For the convenience of the reader we repeat the relevant material. Given sequence $\gamma = (\gamma_s)_{s=1}^\infty$ with $0 < \gamma_s < 1/4$, let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. Define $P_2(x) = x(x-1)$, $P_{2^{s+1}} = P_{2^s}(P_{2^s} + r_s)$ and $E_s = \{x \in \mathbb{R} : P_{2^{s+1}}(x) \le 0\}$ for $s \in \mathbb{N}$. Then $E_s = \cup_{j=1}^{2^s} I_{j,s}$, where the s-th level *basic intervals* $I_{j,s}$ are disjoint and $\max_{1\le j\le 2^s} |I_{j,s}| \to 0$ as $s \to \infty$. Here, $(P_{2^s} + r_s/2)(E_s) = [-r_s/2, r_s/2]$, so the sets E_s are polynomial inverse images of intervals. Since $E_{s+1} \subset E_s$, we have a Cantor-type set $K(\gamma) := \cap_{s=0}^\infty E_s$.

In what follows we will consider only γ satisfying the assumptions

$$\gamma_k \le 1/32 \quad \text{for} \quad k \in \mathbb{N} \quad \text{and} \quad \sum_{k=1}^\infty \gamma_k < \infty. \tag{2}$$

The lengths $l_{j,s}$ of the intervals $I_{j,s}$ of the s-th level are not the same, but, provided (2), we can estimate them in terms of the parameter $\delta_s = \gamma_1\gamma_2\cdots\gamma_s$ ([10], L.6):

$$\delta_s < l_{j,s} < C_0\,\delta_s \quad \text{for} \quad 1 \le j \le 2^s, \tag{3}$$

where $C_0 = \exp(16\sum_{k=1}^\infty \gamma_k)$. Each $I_{j,s}$ contains two *adjacent* basic subintervals $I_{2j-1,s+1}$ and $I_{2j,s+1}$. Let $h_{j,s} = l_{j,s} - l_{2j-1,s+1} - l_{2j,s+1}$ be the distance between them. By Lemma 4 in [10],

$$h_{j,s} > (1 - 4\gamma_{s+1})l_{j,s} \ge 7/8 \cdot l_{j,s} > 7/8 \cdot \delta_s \quad \text{for all } j \le 2^s. \tag{4}$$

In addition, by T.1 in [10], the level domains $D_s = \{z \in \mathbb{C} : |P_{2^s}(z) + r_s/2| < r_s/2\}$ form a nested family and $K(\gamma) = \cap_{s=0}^\infty \overline{D}_s$. The value $R_s = 2^{-s}\log 2 + \sum_{k=1}^s 2^{-k}\log\frac{1}{\gamma_k}$ represents the Robin constant of $\overline{D}_s$. Therefore, the set $K(\gamma)$ is non-polar if and only if $Rob(K(\gamma)) = \sum_{n=1}^\infty 2^{-n}\log\frac{1}{\gamma_n} = \sum_{n=1}^\infty 2^{-n-1}\log\frac{1}{\delta_n} < \infty$.

We decompose all zeros of P_{2^s} into s groups. Let $X_0 = \{x_1, x_2\} = \{0, 1\}, X_1 = \{x_3, x_4\} = \{l_{1,1}, 1 - l_{2,1}\}, \cdots, X_k = \{l_{1,k}, l_{1,k-1} - l_{2,k}, \cdots, 1 - l_{2^k,k}\}$ for $k \le s-1$. Thus, $X_k = \{x : P_{2^k}(x) + r_k = 0\}$ contains all zeros of $P_{2^{k+1}}$ that are not zeros of P_{2^k}. Set $Y_s = \cup_{k=0}^s X_k$. Then $P_{2^s}(x) = \prod_{x_k\in Y_{s-1}}(x - x_k)$. Clearly, $\#(X_s) = 2^s$ for $s \in \mathbb{N}$ and $\#(Y_s) = 2^{s+1}$ for $s \in \mathbb{N}_0$. We refer *s-th type* points to the elements of X_s.

The points from Y_s can be ordered using, as in [8], the *rule of increase of type*. First we take points from X_0 and X_1 in the ordering given above. To put in order the set X_2, for $1 \le j \le 4$, we take x_{j+4} as the point of the second type which is the closest to x_j. Thus, $x_5 = x_1 + l_{1,2}$, $x_6 = x_2 - l_{4,2}, \cdots$ and the ordered set X_2 is $\{x_5, x_7, x_8, x_6\}$. In other words, the ordered set X_2 can be obtained from $X_0 \cup X_1$ if we arrange this set in increasing way and enlarge every index of x by 4. Similarly, each $X_k = \{x_{2^k+1}, \cdots, x_{2^{k+1}}\}$ can be ordered. See [12] for more details.

In the same way, any N points can be chosen on each basic interval. Suppose $2^n \le N < 2^{n+1}$ and the points $Z = (x_{k,j,s})_{k=1}^N$ are chosen on $I_{j,s}$ by this rule. Then Z includes all 2^n zeros of $P_{2^{s+n}}$ on $I_{j,s}$ (points of the type $\le s+n-1$) and some $N - 2^n$ points of the type $s+n$. In what follows, we write $Z = (z_{k,j,s})_{k=1}^N$ or $Z = (z_k)_{k=1}^N$, when no confusion can arise, for the same set in the order of increasing.

We use two technical lemmas from [12]. We suppose that γ satisfies (2).

Let $2^n \le N < 2^{n+1}$ and a basic interval $I_{j,s}$ be given. Suppose $Z_N = (x_{k,j,s})_{k=1}^N$ and $Z_{N+1} = (x_{k,j,s})_{k=1}^{N+1}$ are chosen on $I_{j,s}$ by the rule of increase of type. Write $C_1 = 8/7 \cdot (C_0 + 1)$.

Lemma A. *(Lemma 2.2 from [12]) For each* $x \in \mathbb{R}$ *with* $\mathrm{dist}(x, K(\gamma) \cap I_{j,s})) \le \delta_{s+n}$ *and* $z \in Z_{N+1}$ *we have* $\delta_{s+n} \prod_{k=2}^N d_k(x, Z_N) \le C_1^N \prod_{k=2}^{N+1} d_k(z, Z_{N+1})$.

Let $(z_k)_{k=1}^{N+1}$ be the same set Z_{N+1} but arranged in ascending order. For $q = 2^m - 1$ with $m < n$ and $1 \le j \le N + 1 - q$, let $J = \{z_j, \cdots, z_{j+q}\}$ be 2^m consecutive points from Z_{N+1}. Given j, we consider all possible chains of strict embeddings of segments of natural numbers: $[j, j + q] = [a_0, b_0] \subset [a_1, b_1] \subset \cdots \subset [a_{N-q}, b_{N-q}] = [1, N + 1]$, where $a_k = a_{k-1}$, $b_k = b_{k-1} + 1$ or $a_k = a_{k-1} - 1$, $b_k = b_{k-1}$ for $1 \le k \le N - q$. Every chain generates the product $\prod_{k=1}^{N-q}(z_{b_k} - z_{a_k})$. For fixed J, let $\Pi(J)$ denote the minimum of these products for all possible chains.

Lemma B. *(Lemma 2.3 from [12]) For each* $J \subset Z_{N+1}$ *there exists* $\tilde{z} \in J$ *such that* $\prod_{k=q+2}^{N+1} d_k(\tilde{z}, Z_{N+1}) \le \Pi(J)$.

We will characterize EP of $K(\gamma)$ in terms of the values $B_k = 2^{-k-1} \cdot \log \frac{1}{\delta_k}$ that have Potential Theory meaning: $Rob(K(\gamma)) = \sum_{k=1}^{\infty} B_k$. The main condition is (compare with (3) in [9]):

$$\frac{B_{n+s}}{\sum_{k=s}^{n+s} B_k} \to 0 \quad \text{as} \quad n \to \infty \quad \text{uniformly with respect to} \quad s. \tag{5}$$

We see that this condition allows polar sets.

Example 1. Let $\gamma_1 = \exp(-4B)$ and $\gamma_k = \exp(-2^k B)$ for $k \ge 2$, where $B \ge \frac{1}{4} \log 32$, so (2) is valid. Here, $B_k = B$ for all k. Hence (5) is satisfied and the set $K(\gamma)$ is polar.

The condition (5) means that

$$\forall \varepsilon \, \exists s_0, \, \exists n_0 : \; B_{s+n} < \varepsilon(B_s + \cdots + B_{s+n}) \; \text{ for } \; n \ge n_0, \, s \ge s_0. \tag{6}$$

Clearly, instead of $\exists s_0$ one can take above $\forall s_0$. Let us show that (6) is equivalent to

$$\forall \varepsilon_1 \; \forall m \in \mathbb{N}_0 \exists N : \; B_{s+n-m} + \cdots + B_{s+n} < \varepsilon_1(B_s + \cdots + B_{s+n-m-1}), \, n \ge N, s \ge 1. \tag{7}$$

Indeed, the value $m = 0$ in (7) gives (6) at once. For the converse, remark that in (7) we can take $\varepsilon_1(B_s + \cdots + B_{s+n})$ on the right side, so here we consider (7) in this new form. Suppose (6) is valid. Given ε_1 and m, take $\varepsilon = \varepsilon_1/(m + 1)$ and the corresponding value n_0 from (6). Take $N = n_0 + m$. Then for $n \ge N$ and $0 \le k \le m$ we have $n - k \ge n_0$, so $B_{s+n-k} < \varepsilon(B_s + \cdots + B_{s+n-k}) < \varepsilon(B_s + \cdots + B_{s+n})$. Summing these inequalities, we obtain a new form of (7).

It follows that the negation of the main condition can be written as

$$\exists \varepsilon \; \exists m : \forall N \, \exists n > N : \; \sum_{s+n-m}^{s+n} B_k > \varepsilon \sum_{s}^{s+n-m-1} B_k \; \text{for} \; s = s_j \uparrow \infty. \tag{8}$$

Also, (6) is equivalent to

$$\forall \varepsilon \;\; \exists m, n_0, s_0 : \; B_{s+n} < \varepsilon(B_{s+n-m} + \cdots + B_{s+n-1}) \; \text{ for } n \ge n_0, \, s \ge s_0. \tag{9}$$

Indeed, comparison of right sides of inequalities shows that (9) implies (6). Conversely, given ε, take n_0 such that (6) is valid with $\varepsilon/(1+\varepsilon)$ instead of ε. Take $m = n_0$. Then for $n \geq n_0, s \geq s_0$ we have $\tilde{s} = s+n-m \geq s_0$ and, by (6), $B_{s+n} = B_{\tilde{s}+m} < \frac{\varepsilon}{1+\varepsilon}(B_{\tilde{s}} + \cdots B_{\tilde{s}+m})$, which is (9).

We will use a "geometric" version of (9) in terms of (δ_k)

$$\forall M \ \ \exists m, n_0, s_0 : \ \delta_{s+n-1}\, \delta^2_{s+n-2} \cdots \delta^{2^{m-1}}_{s+n-m} < \delta^M_{s+n} \ \text{ for } n \geq n_0,\ s \geq s_0. \tag{10}$$

4. Extension operator for $\mathcal{E}(K(\gamma))$

Here, as in [2], we use the method of local Newton interpolations. Let K be shorthand for $K(\gamma)$. We fix a nondecreasing sequence of natural numbers $(n_s)_{s=0}^{\infty}$ with $n_s \geq 2$ and $n_s \to \infty$. Given function f on K, we interpolate f at 2^{n_0} points that are chosen by the rule of increase of type on the whole set. Half of points are located on $K \cap I_{1,1}$. We continue interpolation on this set up to the degree 2^{n_1}. Separately we do the same on $K \cap I_{2,1}$. Continuing in this fashion, we interpolate f with higher and higher degrees on smaller and smaller basic intervals. At each step the additional points are chosen by the rule of increase of type. Interpolation on $I_{j,s}$ does not affect other intervals of the same level due to the following function.

Let $t > 0$ and a compact set E on the line be given. Then $u(\cdot, t, E)$ is a C^∞-function with the properties: $u(\cdot, t, E) \equiv 1$ on E, $u(x, t, E) = 0$ for $dist(x, E) > t$ and $\sup_{x\in\mathbb{R}} |u^{(p)}_{x^p}(x, t, K)| \leq c_p\, t^{-p}$, where the constant c_p depends only on p. Let $c_p \nearrow$.

For any interval I and points $(z_k)_{k=1}^{N+1} \subset I$, let $\Omega(x) = \prod_{k=1}^{N+1}(x - z_k), \omega_k(x) = \frac{\Omega(x)}{(x-z_k)\Omega'(z_k)}$ and $L_N(f, x, I) = \sum_{k=1}^{N+1} f(z_k)\, \omega_k(x)$ be the interpolating polynomial with nodes at these points.

We define $N_s = 2^{n_s} - 1$ and $M_s = 2^{n_{s-1}-1} - 1$ for $s \geq 1$, $M_0 = 1$. Now, for fixed s, we take $M_s + 1 \leq N \leq N_s$, so $2^n \leq N < 2^{n+1}$ with $n \in \{n_{s-1} - 1, \cdots, n_s - 1\}$. For such N and s let $t_N := \delta_{s+n}$. Fix j with $1 \leq j \leq 2^s$. Next, we choose the set $Z_{N+1} = (x_{k,j,s})_{k=1}^{N+1} = (z_k)_{k=1}^{N+1}$ on $I_{j,s}$ by the rule of increase of type and consider, for given $f \in \mathcal{E}(K(\gamma))$ and $x \in \mathbb{R}$, the value

$$A_{N,j,s}(f, x) := [L_N(f, x, I_{j,s}) - L_{N-1}(f, x, I_{j,s})]\, u(x, t_N, I_{j,s} \cap K).$$

We call $A_{j,s}(f, x) := \sum_{N=M_s+1}^{N_s} A_{N,j,s}$ the *accumulation sum*. The last term here corresponds to the interpolation on $I_{j,s}$ at 2^{n_s} points. In order to continue interpolation on subintervals of $I_{j,s}$, let us consider the *transition sum*

$$T_{k,s}(f, x) := [L_{M_{s+1}}(f, x, I_{k,s+1}) - L_{N_s}(f, x, I_{j,\,s})]\, u(x, \delta_{s+n_s-1}, I_{k,\,s+1} \cap K),$$

where we suppose $1 \leq k \leq 2^{s+1}$, $j = [\frac{k+1}{2}]$. Of course, $I_{k,s+1} \subset I_{j,\,s}$.

As above, we represent the difference in brackets in the telescoping form:

$$[L_{M_{s+1}} - L_{N_s}] = -\sum_{N=2^{n_s-1}}^{2^{n_s}-1} [L_N(f, x, I_{j,\,s}) - L_{N-1}(f, x, I_{j,\,s})].$$

Here, the interpolating set for L_N consists of $M_{s+1} + 1$ points of $Y_{s+n_s-1} \cap I_{k,s+1}$ and $N - M_{s+1}$ points on $I_{i,s+1}$. The second parameter of u is smaller than the mesh size of Z, so $T_{k,s}(f, x) \neq 0$ only near $I_{k,s+1}$.

Consider a linear operator

$$W(f, \cdot) = L_{M_0}(f, \cdot, I_{1,\,0})\, u(\cdot, 1, K) + \sum_{s=0}^{\infty} \Big[\sum_{j=1}^{2^s} A_{j,s}(f, \cdot) + \sum_{k=1}^{2^{s+1}} T_{k,s}(f, \cdot)\Big]. \tag{11}$$

We remark at the outset that, for fixed $x \in \mathbb{R}$ and s, because of the choice of parameters for the function u, at most one value $A_{j,s}$ does not vanish. The same is valid for $T_{k,s}$.

Let us show that W extends functions from $\mathcal{E}(K)$, provided a suitable choice of $(n_s)_{s=0}^{\infty}$. Define $n_0 = n_1 = 2$ and $n_s = [\log_2 \log \frac{1}{\delta_s}]$ for $s \ge 2$. Then $n_s \le n_{s+1}$ and

$$\frac{1}{2}\log\frac{1}{\delta_s} < 2^{n_s} \le \log\frac{1}{\delta_s} \quad \text{for } s \ge 2. \tag{12}$$

Lemma 4.1. *Let $(n_s)_{s=0}^{\infty}$ be given as above. Then for any $f \in \mathcal{E}(K(\gamma))$ and $x \in K(\gamma)$ we have $W(f,x) = f(x)$.*

Proof. Let us fix a natural number q with $q > 2 + \log(8C_0/7)$, where C_0 is defined in (3). By the telescoping effect,

$$W(f,x) = \lim_{s\to\infty} L_{M_s}(f,x,I_{j,s}), \tag{13}$$

where $j = j(s,x)$ is chosen in such a way that $x \in I_{j,s}$. As in [7],

$$|\,L_{M_s}(f,x,I_{j,s}) - f(x)| \le ||\,f\,||_{\,q} \sum_{k=1}^{2^n} |\,x - z_k|^{\,q}\,|\,\omega_k(x)\,|. \tag{14}$$

Here n is shorthand for $n_{s-1} - 1$ and s is such that $M_s = 2^n - 1 > q$. The interpolating set $(z_k)_{k=1}^{2^n}$ for L_{M_s} consists of all points of the type $\le s+n-1$ on $I_{j,s}$. Given point x, we consider the chain of basic intervals containing it: $x \in I_{j_n,s+n} \subset \cdots \subset I_{j_1,s+1} \subset I_{j,s}$. We see that $I_{j_n,s+n}$ contains one interpolating point, $I_{j_{n-1},s+n-1} \setminus I_{j_n,s+n}$ does one more z_i, $I_{j_{n-2},s+n-2} \setminus I_{j_{n-1},s+n-1}$ contains two such points, etc. Thus, for fixed k, we get

$$|\,x - z_k|^{\,q} \prod_{i=1, i\ne k}^{2^n} |x - z_i| \le l_{j,s}^{q-1} \cdot l_{j_n,s+n} \cdot l_{j_{n-1},s+n-1} \cdot l_{j_{n-2},s+n-2}^2 \cdots l_{j,s}^{\,2^{\,n-1}}.$$

By (3), this does not exceed $C_0^{2^n+q-1}\delta_{s+n}\,\delta_{s+n-1}\,\delta_{s+n-2}^2 \cdots \delta_{s+1}^{2^{n-2}}\;\delta_s^{2^{n-1}+q-1}$.

On the other hand, by a similar argument, for the denominator of $|\,\omega_k(x)\,|$ we have

$$|\,z_k - z_1|\cdots|\,z_k - z_{k-1}|\,\cdot|\,z_k - z_{k+1}|\cdots|\,z_k - z_{2^n}| \ge l_{q_{n-1},s+n-1} \cdot h_{q_{n-2},s+n-2}^2 \cdots h_{j,s}^{\,2^{\,n-1}}$$

for some indices $q_{n-1}, q_{n-2}, \cdots$. The last product exceeds $(7/8)^{2^n-2}\delta_{s+n-1}\,\delta_{s+n-2}^2 \cdots \delta_s^{2^{n-1}}$, by (4). It follows that

$$\text{LHS of (14)} \le ||\,f\,||_{\,q}\, 2^n\, C_0^{q-1} (8C_0/7)^{\,2^{\,n}}\; \delta_{s+n}\,\delta_s^{q-1}.$$

The expression on the right side approaches zero as $s \to \infty$. Indeed, $2^n < \log(1/\delta_{s-1})$, by (12), and $2^n(8C_0/7)^{\,2^{\,n}}\;\delta_s^{q-1} < 1$ due to the choice of q. Thus the limit in (13) exists and equals $f(x)$. □

5. Extension property of weakly equilibrium Cantor-type sets

We need two more lemmas.

Lemma 5.1. *Let γ satisfy (2), $q = 2^m$, $r = 2^n$ with $m < n$ and $Z = (z_k)_{k=1}^r$ with $z_1 < \cdots < z_r$ be all points of the type $\le s+n-1$ on $I_{1,s}$ for some $s \in \mathbb{N}_0$. Let $f(x) = \prod_{k=1}^r (x - z_k)$ for $x \in K(\gamma) \cap I_{1,s}$ and $f = 0$ on $K(\gamma) \setminus I_{1,s}$. Then $|f|_{0,K(\gamma)} \le C_0^r \cdot \delta_{n+s} \cdot \delta_{n+s-1} \cdot \delta_{n+s-2}^2 \cdots \delta_s^{2^{n-1}}$, $|f^{(q)}(0)| \ge q! \cdot (7/8)^{r-q} \cdot \delta_{n+s-m-1}^{2^m} \cdots \delta_s^{2^{n-1}}$ and $||f||_r \le 2 \cdot r!$.*

Proof. Fix $\tilde{x}$ that realizes $|f|_{0,K(\gamma)}$ and a chain of basic intervals containing this point: $\tilde{x} \in I_{j_0,n+s} \subset I_{j_1,n+s-1} \subset \cdots \subset I_{j_n,s} = I_{1,s}$. Arguing as in Lemma 4.1, we see that $|f|_{0,K(\gamma)} \leq l_{j_0,n+s} \cdot l_{j_1,n+s-1} \cdot l^2_{j_2,n+s-2} \cdots l^{2^{n-1}}_{1,s}$, which, by (3), gives the desired bound.

In order to estimate $|f^{(q)}(0)|$, let us remark that $f^{(q)}(x)$ is a sum of $\binom{r}{q}$ products, each product has a coefficient $q!$ and consists of $r-q$ terms $(x - z_k)$. One of these products is $g(x) := \prod_{k=q+1}^{r}(x - z_k)$. All products are nonnegative at $x = 0$, since $r-q$ is even. From here, $|f^{(q)}(0)| \geq q! \cdot g(0)$. Taking into account the location of points from Z, we get $g(0) = \prod_{k=q+1}^{r} z_k > h^{2^m}_{1,n+s-m-1} \cdots h^{2^{n-1}}_{1,s} > (7/8)^{r-q} \cdot \delta^{2^m}_{n+s-m-1} \cdots \delta^{2^{n-1}}_{s}$, by (4). The bound of $||f||_r$ is evident. □

In the next Lemma, for given $2^n \leq N < 2^{n+1}$, we consider $\Omega_N(x) = \prod_{k=1}^{N}(x - z_k)$ with $Z_N = (z_k)_{k=1}^{N}$, where the points are chosen on $I_{j,s}$ by the rule of increase of type. Let $u(x) = u(x, \delta_{s+n}, I_{j,s} \cap K(\gamma))$.

Lemma 5.2. *The bound* $|(\Omega_N \cdot u)^{(p)}(x)| \leq 2^p\,(C_0 + 1)\,c_p\,\delta_{s+n}^{-p+1}\,N^p \prod_{k=2}^{N} d_k(x, Z_N)$ *is valid for each* $p < N$ *and* $x \in \mathbb{R}$.

Proof. By Leibnitz's formula, $|(\Omega_N \cdot u)^{(p)}(x)| \leq \sum_{i=0}^{p} \binom{p}{i} |\,\Omega_N^{(i)}(x)|\, c_{p-i}\delta_{s+n}^{-p+i}$. Since d_k increases, we have $|\Omega_N^{(i)}(x)| \leq \frac{N!}{(N-i)!} \prod_{k=i+1}^{N} d_k(x, Z_N)$. This gives

$$|(\Omega_N \cdot u)^{(p)}(x)| \leq 2^p\, c_p\, \delta_{s+n}^{-p} \cdot \max_{0 \leq i \leq p}\, (N\,\delta_{s+n})^i \prod_{k=i+1}^{N} d_k(x, Z_N). \tag{15}$$

The set Z_N consists of 2^n endpoints of subintervals of the level $s+n-1$ covered by $I_{j,s}$ and $N - 2^n$ points of the type $s+n$. Here, $\mathrm{dist}(x, I_{j,s} \cap K) = |x - x_0| \leq \delta_{s+n}$ for some x_0. Let $x_0 \in I_{i,s+n} \subset I_{m,s+n-1}$. Then $I_{m,s+n-1}$ contains from 2 to 4 points of Z_N. In all cases, $d_1(x, Z_N) \leq l_{i,s+n} + \delta_{s+n} \leq (C_0+1)\delta_{s+n}$, by (3). Also, $\delta_{s+n}/2 \leq d_2 \leq (C_0+1)\delta_{s+n-1}$. Here the lower bound corresponds to the case $\#(I_{i,s+n} \cap Z_N) = 2$, whereas the upper bound deals with $\#(I_{m,s+n-1} \cap Z_N) = 2$. Similarly, $d_3 \geq h_{m,s+n-1} - \delta_{s+n}$. From (4) and (2) it follows that $d_3 \geq 7/8\ \delta_{s+n-1} - \delta_{s+n} \geq 27\,\delta_{s+n}$. This gives $\delta_{s+n}^{i-1} d_{i+1} \cdots d_N \leq (C_0+1) d_2 \cdots d_N$ for $0 \leq i \leq p$ and, by (15), the lemma follows. □

We can now formulate our main result.

Theorem 5.3. *Suppose* γ *satisfies* (2). *Then* $K(\gamma)$ *has the extension property if and only if* (5) *is valid.*

Proof. Recall that the extension property of a set is equivalent to the condition (DN) of the corresponding Whitney space. Due to L. Frerick [3, Prop. 3.8], $\mathcal{E}(K)$ satisfies (DN) if and only if for any $\varepsilon > 0$ and for any $q \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and $C > 0$ such that $|\cdot|_q^{1+\varepsilon} \leq C|\cdot|_{0,K}\ \ ||\cdot||_r^{\varepsilon}$. Hence, in order to prove that (5) is necessary for EP of $K(\gamma)$, we can show that (8) implies the lack of (DN) for $\mathcal{E}(K(\gamma))$, that is there exist $\varepsilon > 0$ and q such that for any $r \in \mathbb{N}$ one can find a sequence $(f_j) \subset \mathcal{E}(K(\gamma))$ with

$$|\,f_j|_q^{1+\varepsilon}\,|\,f_j|_{0,K(\gamma)}^{-1}\,||\,f_j||_r^{-\,\varepsilon} \to \infty \quad \text{as} \quad j \to \infty.$$

Let us fix ε and m from the condition (8) and take $q = 2^m$. For each fixed large r (clearly, we can take it in the form $r = 2^n$) and s_j defined by (8), we consider the function f_j given in Lemma 5.1 for $s = s_j$. Then

$$C\,|\,f_j|_q^{1+\varepsilon}\,|\,f_j|_{0,K(\gamma)}^{-1}\ ||\,f_j||_r^{-\,\varepsilon} \geq (\delta_{n+s} \cdot \delta_{n+s-1} \cdot \delta^2_{n+s-2} \cdots \delta^{2^{m-1}}_{n+s-m})^{-1} (\delta^{2^m}_{n+s-m-1} \cdots \delta^{2^{n-1}}_{s})^{\varepsilon},$$

where C does not depend on j. The right side here goes to infinity. Indeed, its logarithm is $2^{n+s}\,\{2B_{n+s} + B_{n+s-1} + \cdots + B_{n+s-m} - \varepsilon[B_{n+s-m-1} + \cdots + B_s]\}$ and the expression in braces exceeds B_{n+s} by (8).

Therefore, the whole value exceeds $2^{n+s}B_{n+s} = \frac{1}{2}\log\frac{1}{\delta_{s+n}}$, which goes to infinity when $s = s_j$ increases. Thus, EP of $K(\gamma)$ implies (5).

For the converse, we consider the extension operator W from Section 4, where (n_s) are chosen as in (12). We proceed to show that W is bounded provided (10), which is equivalent to (5). Let us fix any natural number p. This p and C_1 from Lemma A define $M = 2p + 2 + \log(2C_1)$. We fix $m \in \mathbb{N}$ that corresponds to M in the sense of (10). Let $q = 2^m - 1$ and $r = r(q)$ be defined by (1). We will show that the bound $|(W(f,x))^{(p)}| \le C\,||f||_r$ is valid for some constant $C = C(p)$ and all $f \in \mathcal{E}(K)$, $x \in \mathbb{R}$.

Given f and x, let us consider terms of accumulation sums. For fixed $s \in \mathbb{N}$ we choose $j \le 2^s$ such that $x \in I_{j,s}$. Fix N with $2^n \le N < 2^{n+1}$ for $n_{s-1} - 1 \le n \le n_s - 1$, so $M_s + 1 \le N \le N_s$. For large enough s the value N exceeds p and q. We take Z_N and Z_{N+1}, as in Lemma A. By Newton's representation of interpolating operators in terms of divided differences, we have

$$A_{N,j,s}(f,x) = [z_1, \cdots, z_{N+1}]f \cdot \Omega_N(x)\,u(x),$$

where Ω_N and u are taken as in Lemma 5.2. We aim to show that

$$N_s\,|A^{(p)}_{N,j,s}(f,x)| \le s^{-2}\,||f||_r \tag{16}$$

for large s. This gives convergence of the accumulation sums.

For the divided difference we use (4) from [2]:

$$|\,[z_1, \cdots, z_{N+1}]f\,| \le 2^{N-q}\,|||f|||_q\,(\Pi(J_0))^{-1}, \tag{17}$$

where $\Pi(J_0) = min_{1\le j\le N+1-q}\Pi(J)$ for $\Pi(J)$ defined in Lemma B. Fix $\tilde{z} \in J_0$ that corresponds to this set in the sense of Lemma B.

Applying Lemma 5.2 and Lemma A for $z = \tilde{z}$ yields

$$|(\Omega_N \cdot u)^{(p)}(x)| \le C\,\delta_{s+n}^{-p}\,N^p\,C_1^N \prod_{k=2}^{N+1} d_k(\tilde{z}, Z_{N+1}) \text{ with } C = 2^p(C_0+1)\,c_p.$$

On the other hand, (17) and Lemma B for J_0 give

$$|\,[z_1, \cdots, z_{N+1}]f\,| \le 2^{N-q}\,|||f|||_q \prod_{k=q+2}^{N+1} d_k^{-1}(\tilde{z}, Z_{N+1}).$$

Combining these we see that

$$|A^{(p)}_{N,j,s}(f,x)| \le C\;|||f|||_q\,\delta_{s+n}^{-p}\,N^p\,(2C_1)^N \prod_{k=2}^{q+1} d_k(\tilde{z}, Z_{N+1}). \tag{18}$$

Recall that the set Z_{N+1} includes all points of the type $\le s+n-1$ on $I_{j,s}$ and $N+1-2^n$ points of the type $s+n$. We can only enlarge the product $\prod_{k=2}^{q+1} d_k(\tilde{z}, Z_{N+1})$ if we will consider only distances from $\tilde{z}$ to points from $Y_{s+n-1} \cap I_{j,s}$. Arguing as in Lemma 4.1, we get $\prod_{k=2}^{q+1} d_k(\tilde{z}, Z_{N+1}) \le C_0^q\delta_{s+n-1}\,\delta_{s+n-2}^2 \cdots \delta_{s+n-m}^{2^{m-1}}$. We observe that $d_1(\tilde{z}, Z_{N+1}) = 0$ is not included into the product on the left side. By (10), $\prod_{k=2}^{q+1} d_k(\tilde{z}, Z_{N+1}) \le C_0^q\;\delta_{s+n}^M$.

In order to get (16), it is enough to show that

$$s^2\;N_s\,N^p\,(2C_1)^N\,\delta_{s+n}^{M-p} \to 0 \text{ as } s \to \infty. \tag{19}$$

Here, by (12), $N_s\,N^p\,<\,2^{n_s(p+1)}\,\leq\,\log(1/\delta_s)^{p+1}\,<\,\delta_s^{-p-1}$. Also, $(2C_1)^N\,<\,\delta_s^{-\log(2C_1)}$. Clearly, we can replace δ_{s+n} in (19) by δ_s. Then, because of the choice of M, the product in (19) does not exceed $s^2\,\delta_s$, which approaches 0 as $s \to \infty$, since, by (2), $\delta_s \leq 32^{-s}$.

Similar arguments are used for terms of the transition sums. □

6. Extension of basis elements

An extension operator for the spaces $\mathcal{E}(K(\gamma))$ can also be constructed by means of suitable extensions of basis elements of the space. It is interesting that for sufficiently small sets with EP both approaches coincide.

Let $e_0 \equiv 1$ and $e_N(x) = \prod_1^N (x - x_k)$ for $N \in \mathbb{N}$, where the points $(x_k)_1^\infty$ are chosen on $K(\gamma)$ by the rule of increase of type. Then, by Theorem 3.3 in [12], $(e_N)_{N=0}^\infty$ is a Schauder basis in $\mathcal{E}(K(\gamma))$, provided

$$\forall Q\ \exists m, k_0 : \quad Q \leq B_{k-m} + \cdots + B_k \quad \text{for} \quad k \geq k_0. \tag{20}$$

Thus, in this case, the space possesses a strict polynomial basis. If, in addition,

$$\forall M, Q\ \exists m, k_0 : \quad Q + M \cdot B_k \leq B_{k-m} + \cdots + B_k \quad \text{for} \quad k \geq k_0,$$

then one can take simultaneous (suitable for all norms) extensions $\tilde{e}_N = e_N \cdot u(\cdot, \delta_n, K(\gamma))$, where $2^n \leq N < 2^{n+1}$. The biorthogonal functionals are given by the divided differences $\xi_N(f) = [x_1, x_2, \cdots, x_{N+1}]f$. Here, the Mityagin method gives the operator $W(f) = \sum_{n=0}^\infty \xi_n(f) \cdot \tilde{e}_N$ for $f = \sum_{n=0}^\infty \xi_n(f) \cdot e_n$. In the notations of Section 4, $W(f) = L_0(f, \cdot, I_{1,\,0})\,u(\cdot, 1, K(\gamma)) + \sum_{n=1}^\infty A_{N,1,0}(f, \cdot)$, which is exactly the Pawłucki–Pleśniak operator, if $(x_k)_1^N$ are the Fekete points on the set. We conjecture that, at least for $N = 2^n$, this is the case.

In general, without (20), $(e_N)_{N=0}^\infty$ does not have the basis property. Here a basis can be constructed by means of local interpolations. The condition (5) provides existence of extensions of basis elements that correspond to the accumulation sums in (11). However, the terms of transition sums do not have simple representations in terms of such extensions.

7. Two examples

First we consider regular sequences $(B_k)_{k=1}^\infty$. Let $\beta_k = (\log B_k)/k$. We say that $(B_k)_{k=1}^\infty$ is *regular* if, for some k_0, both sequences $(B_k)_{k=k_0}^\infty$ and $(\beta_k)_{k=k_0}^\infty$ are monotone. Recall that $(B_k)_{k=1}^\infty$ has *subexponential growth* if $\beta_k \to 0$ as $k \to \infty$.

For example, given $a > 1$, let $\gamma_k^{(1)} = k^{-a}, \gamma_k^{(2)} = a^{-k}, \gamma_k^{(3)} = \exp(-a^k)$ for large enough k. Then $\gamma^{(j)}$ for $1 \leq j \leq 3$ generate regular $B^{(j)}$ with $B_k^{(1)} \sim 2^{-k-1}\,a\,k\log k, B_k^{(2)} \sim 2^{-k-2}\,k^2\,\log a$, $B_k^{(3)} \sim (a/2)^{k+1}/(a-1)$. Here, $\beta_k^{(1)}$, $\beta_k^{(2)} \nearrow -\log 2$ and $\beta_k^{(3)} \to -\log(a/2)$, so $B^{(j)}$ are not of subexponential growth, except $B^{(3)}$ for $a = 2$. We see that (5) is valid in the first two cases and in the third case with $a \leq 2$.

More generally, (5) is valid for each monotone convergent $(B_k)_{k=1}^\infty$. Indeed, if $B_k \searrow B \geq 0$, then LHS of (5) does not exceed $(n+1)^{-1}$. If $B_k \nearrow B$, then we take s_0 with $B_s > B/2$ for $s \geq s_0$. Then $B_s + \cdots B_{s+n} \geq (n+1)B/2$ and LHS of (5) $< 2(n+1)^{-1}$. This covers the case of regular sequences $(B_k)_{k=1}^\infty$ when β_k are negative. Let us show that (5) is valid as well for divergent regular sequences $(B_k)_{k=1}^\infty$ of subexponential growth.

Theorem 7.1. *Let $(B_k)_{k=1}^\infty$ be regular with positive values of β_k. Then (5) is valid if and only if $(B_k)_{k=1}^\infty$ has subexponential growth.*

Proof. A regular sequence $(B_k)_{k=1}^\infty$ is not of subexponential growth, provided $\beta_k > 0$, in the following three cases: $\beta_k \nearrow \beta < \infty$, $\beta_k \nearrow \infty$ and $\beta_k \searrow \varepsilon_0 > 0$. We aim to show that (5) is not valid under the circumstances.

In the first case, given s and n, let $b = \exp \beta_{s+n}$. Then $b - 1 \geq \exp \beta_1 - 1 > \beta_1 > 0$ and $b \leq \exp \beta$. Here, $\sum_{k=s}^{s+n} B_k < b^{s+n+1}/(b-1)$ as $B_k = \exp(k\beta_k) \leq b^k$ for such k. Therefore, $B_{s+n}/\sum_{k=s}^{s+n} B_k > (b-1)/b > \beta_1/\exp \beta$, which contradicts (5).

If $\beta_k \nearrow \infty$ then, by the same argument, $B_{s+n}/\sum_{k=s}^{s+n} B_k > (b-1)/b > 1/2$ for $s \geq s_0$, where s_0 is fixed with $\exp \beta_{s_0} > 2$.

Suppose $\beta_k \searrow \varepsilon_0$. We fix indices $s_1 < s_2 < \cdots$ such that the intervals I_j connecting points (s_j, β_{s_j}) and $(s_{j+1}, \beta_{s_{j+1}})$ form a convex envelope of the set (k, β_k) on the plane. We start from $s_1 = \max\{s : \beta_s = \beta_1\}$. If s_j is chosen, then we take s_{j+1} with he property: for each k with $s_j \leq k \leq s_{j+1}$ the point (k, β_k) is not over I_j. At any step we can take the next value so large that the slopes of I_j increases to zero. In addition, given s_j, we take s_{j+1} such that

$$(4 - 2\, s_j/s_{j+1})\beta_{s_{j+1}} \geq (3 - s_j/s_{j+1})\beta_{s_j}, \tag{21}$$

which is possible as β_k decreases to a positive limit.

For fixed j, we take $s = s_j$ and $s + n = s_{j+1}$. Let $\tilde{\beta}_k = ak + b$ with $a = -(\beta_s - \beta_{s+n})/n$ and $b = \beta_s + (\beta_s - \beta_{s+n})\, s/n$ for $s \leq k \leq s+n$, so the points $(k, \tilde{\beta}_k)$ are located just on the interval I_j. Also, let $g(x) = ax^2 + bx$ and $\tilde{B}_k = \exp g(k) = \exp(k\, \tilde{\beta}_k)$ on $[s, s+n]$. Of course, $\tilde{B}_s = B_s$ and $\tilde{B}_{s+n} = B_{s+n}$.

It is easy to check that the function g increases on this interval. Hence, $\sum_{k=s}^{s+n} B_k \leq \sum_{k=s}^{s+n} \tilde{B}_k < \int_s^{s+n} g(x)\, dx + B_{s+n}$. By integration by parts, $\int_s^{s+n} g(x)\, dx = g(n+s) \cdot [2a(n+s)+b]^{-1} - g(s) \cdot [2as+b]^{-1} + 2a \int_s^{s+n} g(x)(2ax+b)^{-2} dx$. We neglect the last term, as $a < 0$, and the second term, as $2as + b = g'(s) > 0$. Also, $2a(n+s) + b = (2 + s/n)\beta_{s+n} - (1 + s/n)\beta_s \geq \beta_s/2 \geq \varepsilon_0/2$, by (21). Hence $\int_s^{s+n} g(x)\, dx < 2\, B_{s+n}/\varepsilon_0$ and $B_{s+n}/\sum_{k=s}^{s+n} B_k > \varepsilon_0/(2 + \varepsilon_0)$, so (5) is not valid.

We proceed to show that (5) is valid for $\beta_k \searrow 0$, that is in the case of subexponential growth of $(B_k)_{k=1}^\infty$. Here, for fixed large s and n, we estimate $\sum_{k=s}^{s+n} B_k$ from below. Let $b = \exp \beta_{s+n}$. Then $B_k \geq b^k$ for $s \leq k \leq s+n$. Therefore, $B_{s+n}/\sum_{k=s}^{s+n} B_k \leq \frac{b^n(b-1)}{b^{n+1}-1}$.

If $b^n < 2$ for the given s and n then $b^{n+1} - 1 > (n+1)\beta_{s+n}$. On the other hand, $\exp \beta_{s+n} - 1 < 2\, \beta_{s+n}$ for $\beta_{s+n} < 1$. Thus the fraction above does not exceed $4/(n+1)$.

Otherwise, $b^n \geq 2$ and $b^n < 2(b^{n+1} - 1)$. Here the fraction does not exceed $4\, \beta_{s+n}$. It follows that $B_{s+n}/\sum_{k=s}^{s+n} B_k \leq \max\{4/(n+1), 4\, \beta_n\}$, which is the desired conclusion. □

Our next objective is to consider irregular sequences $(B_k)_{k=1}^\infty$ (compare with Ex.6 in [11]). Given two sequences, $(k_j)_{j=1}^\infty \subset \mathbb{N}$ with $k_{j+1} - k_j \nearrow \infty$ and $(\varepsilon_j)_{j=1}^\infty$ with $\varepsilon_j \searrow 0$, let $\gamma_k = (k+5)^{-2}$ for $k \neq k_j$ and $\gamma_{k_j} = (k_j + 5)^{-2}\varepsilon_j$. Then γ satisfies (2) with $\delta_k = (5!/(k+5)!)^2\, \varepsilon_1 \varepsilon_2 \cdots \varepsilon_j$ for $k_j \leq k < k_{j+1}$. Let $A_j := \log \frac{1}{\varepsilon_1 \varepsilon_2 \cdots \varepsilon_j}$. We will consider only sequences with the property

$$k_{j+1}^2 \cdot A_j^{-1} \to 0 \text{ as } j \to \infty. \tag{22}$$

Provided this condition, $B_k = 2^{-k} \log \frac{(k+5)!}{5!} + 2^{-k-1} A_j \sim 2^{-k-1} A_j$ for $k_j \leq k < k_{j+1}$. In addition, an easy computation shows that for large j,

$$B_{k_j} + B_{k_j+1} + \cdots + B_{k_{j+1}-1} < 3\, B_{k_j}. \tag{23}$$

Now we can construct different examples of compact sets $K(\gamma)$ without extension property.

Example 2. Let $A_j = 2^{k_j}$, so $\varepsilon_j = \exp(-2^{k_j} + 2^{k_{j-1}})$ for $j \geq 2$ and $\varepsilon_1 = \exp(-2^{k_1})$. In this case, (22) is valid under mild restriction $2^{-k_j}\, k_{j+1}^2 \to 0$ as $j \to \infty$. Let us take $s = k_j, n = k_{j+1} - k_j$. Then $B_{s+n} >$

$2^{-k_{j+1}-1} A_{j+1} = 1/2$ and, by (23), $B_s + \cdots + B_{s+n-1} < 3\, B_s < 4 \cdot 2^{-k_j - 1} A_j = 2$. This gives (8) with $\varepsilon = 1/4$ and $m = 0$.

8. Extension property of $K(\gamma)$ and Hausdorff measures

From now on, h is a *dimension function*, which means that $h : (0, T) \to (0, \infty)$ is continuous, nondecreasing and $h(t) \to 0$ as $t \to 0$. The h-Hausdorff content of $E \subset \mathbb{R}$ is defined as

$$M_h(E) = \inf\{\sum h(|G_i|) : E \subset \cup G_i\}$$

and the h-Hausdorff measure of E is

$$\Lambda_h(E) = \liminf_{\delta \to 0}\{\sum h(|G_i|) : E \subset \cup G_i, |G_i| \le \delta\}.$$

Here we consider finite or countable coverings of E by intervals (open or closed).

It is easily seen that $M_h(E) = 0$ if and only if $\Lambda_h(E) = 0$. We write $h_1 \prec h_2$ if $h_1(t) = o(h_2(t))$ as $t \to 0$. Let $h_1 \approx h_2$ if $C^{-1} h_1(t) \le h_2 \le C h_1(t)$ for some constant $C \ge 1$ and $0 < t \le t_0 < T$. We will denote by h_0 the function $h_0(t) = (\log \frac{1}{t})^{-1}$ with $0 < t < 1$, which defines the logarithmic measure of sets.

A set E is called *dimensional* if there is at least one dimension function h that makes E an h-set, that is $0 < \Lambda_h(E) < \infty$. In our case, the set $K(\gamma)$ is dimensional. In [1], following Nevanlinna [15], the corresponding dimension function was presented. Let $\eta(\delta_k) = k$ for $k \in \mathbb{Z}_+$ with $\delta_0 := 1$ and $\eta(t) = k + \log \frac{\delta_k}{t} / \log \frac{\delta_k}{\delta_{k+1}}$ for $\delta_{k+1} < t < \delta_k$. Then $h(t) := 2^{-\eta(t)}$ for $0 < t \le 1$. Clearly, $h(\delta_k) = 2^{-k}$.

Lemma 8.1. *Let γ satisfies (2) and h be defined as above. Then $\Lambda_h(K(\gamma)) = 1$.*

Proof. Take $t = C_0\, \delta_k$, where C_0 is given in (3). Then $\delta_k < t = C_0\, \gamma_k\, \delta_{k-1} < \delta_{k-1}$ for large enough k. Here, $\eta(t) = k - \log C_0 / \log(1/\gamma_k)$ and $h(t) = 2^{-k}\, a_k$ with $a_k := \exp \frac{\log C_0 \cdot \log 2}{\log(1/\gamma_k)}$. Since $\gamma_k \to 0$, given $\varepsilon > 0$, there is k_0 such that $a_k < 1 + \varepsilon$ for $k \ge k_0$. From (3) it follows that $1 = 2^k\, h(\delta_k) < \sum_{j=1}^{2^k} h(l_{j,k}) < 2^k\, h(t) < 1 + \varepsilon$ provided that $k \ge k_0$. Of course, $\Lambda_h(K(\gamma)) \le \sum_{j=1}^{2^k} h(l_{j,k})$ for each k. Since ε is arbitrary, we get $\Lambda_h(K(\gamma)) \le 1$.

Let us show that $\Lambda_h(K(\gamma)) \ge 1$. Fix $\varepsilon > 0$ and choose k_0 such that

$$\varepsilon \log 1/\gamma_k > -\log 2 \cdot \log(1 - 4\gamma_k) \text{ for } k \ge k_0. \tag{24}$$

This can be done as $\gamma_k \to 0$. Take any open covering $\cup G_i$ of $K(\gamma)$. Given ε, we can consider coverings only with $|G_i| < \delta_{k_0}$ for each i. We choose a finite subcover $\cup_{i=1}^{N} G_i$ of $K(\gamma)$.

Fix $i \le N$ and k with $\delta_{k+1} < |G_i| \le \delta_k$. By (3) and (4), the distance between any two basic intervals from E_{k+1} exceeds $(1 - 4\gamma_{k+1})\delta_k$. If $|G_i| < (1 - 4\gamma_{k+1})\delta_k$ then G_i can intersect at most one interval from E_{k+1}. In this case we can consider only $|G_i| \le \max_{1 \le j \le 2^{k+1}} l_{j,k+1} \le C_0 \delta_{k+1}$, by (3). Thus there are two possibilities: $\delta_{k+1} < |G_i| \le C_0 \delta_{k+1}$ or $(1 - 4\gamma_{k+1})\delta_k < |G_i| \le \delta_k$.

In the first case we have $h(|G_i|) > 2^{-k-1}$. Here, G_i intersects at most one interval from E_{k+1} and, by construction, at most 2^{m-k-1} interval from E_m for $m > k$. In turn, in the latter case, $h(|G_i|) > 2^{-k}(1 - \varepsilon)$. Indeed, here, $\eta(|G_i|) < k - \log(1 - 4\gamma_{k+1}) / \log(1/\gamma_{k+1})$ and $h(|G_i|) > 2^{-k}\, a$, where $a = \exp \frac{\log(1-4\gamma_{k+1}) \cdot \log 2}{\log(1/\gamma_{k+1})} > (1 - \varepsilon)$, by (24). Now G_i intersects at most two interval from E_{k+1} and at most 2^{m-k} interval from E_m.

Let us choose m so large that each basic interval from E_m belongs to some G_i. We decompose all intervals from E_m into two groups corresponding to the cases considered above. Counting intervals gives $2^m \le \sum_i' 2^{m-k-1} + \sum_i'' 2^{m-k} < 2^m [\sum_i' h(|G_i|) + \sum_i'' h(|G_i|)(1 - \varepsilon)^{-1}]$. From this we see that $\sum_i h(|G_i|) > 1 - \varepsilon$, which is the desired conclusion, as ε and (G_i) here are arbitrary. □

The same reasoning applies to a part of $K(\gamma)$ on each basic interval.

Corollary 8.2. *Let* γ *and* h *be as in Proposition above,* $k \in \mathbb{N}, 1 \le j \le 2^k$*. Then* $\Lambda_h(K(\gamma) \cap I_{j,k}) = 2^{-k}$*.*

Theorem 8.3. *Suppose* γ *satisfies* (2)*,* h *is defined as above and* $K(\gamma)$ *is not polar. Then* $\mu_K = \Lambda_h|_{K(\gamma)}$*.*

Proof. Here, by Corollary 3.2 in [1], $\mu_{K(\gamma)}(I_{j,k}) = 2^{-k}$, so the values of $\mu_{K(\gamma)}$ and the restriction of Λ_h on $K(\gamma)$ coincide on each basic interval. From here, by Lemma 3.3 in [1], these measures are equal on $K(\gamma)$. □

Thus, a non-polar set $K(\gamma)$ satisfying (2) is indeed *equilibrium* Cantor-type set if we accept for definition of this concept the condition $\mu_K = \Lambda_h|_{K(\gamma)}$, which is more natural than the definition suggested in [10], Section 6.

We recall that there is no complete characterization of polarity of compact sets in terms of Hausdorff measures, see e.g. Chapter V in [15]. On the one hand, a set is polar if its logarithmic measure is finite. This defines a zone Z_{pol} in the scale of growth rate of dimension functions consisting of h with $\liminf_{t\to 0} h(t)/h_0(t) > 0$. If $h \in Z_{pol}$ and $\Lambda_h(K) < \infty$ then $Cap(K) = 0$. On the other hand, functions with $\int_0 h(t)/t\, dt < \infty$ form a non-polar zone Z_{np}: if $h \in Z_{np}$ and $\Lambda_h(K) > 0$ then $Cap(K) > 0$. But, by Ursell [22], the remainder makes up a zone Z_u of uncertainty. One can take two functions in this zone with $h_2 \prec h_1$ and sets K_1, K_2, where K_j is a h_j-set, such that K_2 is polar, K_1 is not, though in the sense of Hausdorff measure the set K_2 is larger than K_1. Indeed, $\Lambda_{h_2}(K_2) > 0$, but $\Lambda_{h_2}(K_1) = 0$ or $\Lambda_{h_1}(K_2) = \infty$, but $\Lambda_{h_1}(K_1) < \infty$.

Let us show that a similar circumstance is valid with the extension property.

Proposition 8.4. *There are two dimension functions* $h_2 \prec h_1$ *and two sets* K_1, K_2*, where* K_j *is an* h_j*-set for* $j \in \{1, 2\}$*, such that the smaller set* K_1 *has the extension property, whereas the larger set* K_2 *does not.*

Proof. Take K_1 from Example 1. Let us show that the corresponding function $h_1 = 2^{-\eta_1}$ is equivalent to h_0. It is enough to find $C > 0$ such that $\eta_0(t) - C \le \eta_1(t) \le \eta_0(t) + C$ for small t. Here, $\eta_0(t) = (\log\log 1/t)/\log 2$, so $h_0(t) = 2^{-\eta_0(t)}$. For the set K_1 we have $\delta_k = \exp(-2^{k+1}B)$ and $\eta_0(\delta_k) = k + \log 2B/\log 2$. If $\delta_{k+1} < t \le \delta_k$ for some k, then $k \le \eta_1(t) < k+1$ and $k + \log 2B/\log 2 \le \eta_0(t) < k + 1 + \log 2B/\log 2$, which gives $h_1 \approx h_0$.

In turn, let K_2 be as in Example 2 with $A_j = 2^{k_j}\, 2^{-j}$ and $\varepsilon_j = \exp(-A_j + A_{j+1})$ for $j \ge 2$. Here we suppose that $(k_j)_{j=1}^{\infty}$ satisfies $2^{-k_j} 2^j\, k_{j+1}^2 \to 0$ as $j \to \infty$. Then (22) and (23) are valid, which, as in Example 2, gives the lack of the extension property for K_2. Let us show that $h_2 \prec h_0$. It is enough to check that $\eta_2(t) - \eta_0(t) \to \infty$ as $t \to 0$. Let $\delta_k < t \le \delta_{k-1}$ with $k_j \le k < k_{j+1}$ for large enough j. Then $\log 1/\delta_k = 2\log((k+5)!/5!) + A_j < 2\,A_j$ and $\eta_0(t) < \eta_0(\delta_k) < k_j + 1 - j$. On the other hand, $\eta_2(t) \ge \eta_2(\delta_{k-1}) = k - 1 \ge k_j - 1$. Therefore, $\eta_2(t) - \eta_0(t) > j - 2$, which completes the proof. □

One can suppose that, at least for the considered family of sets, the scale of growth rate of dimension functions can be decomposed as above into three zones. If $K(\gamma)$ is an h-set for a function h with moderate growth then the set has EP. If the corresponding function h is large enough, then EP fails. Proposition above shows that the zone of uncertainty here is not empty.

We see that $h = h_0$ is not the largest function which allows EP for h-sets $K(\gamma)$. If we take $B_k \nearrow \infty$ of subexponential growth, as in the regular case, then $\delta_k = \exp(-2^{k+1}B_k)$ and $h_0(\delta_k) = 2^{-k-1}B_k^{-1}$, which is essentially smaller than $h(\delta_k) = 2^{-k}$ for the corresponding function h.

Example 3. Let $\log_{(m)} t$ denote the m-th iteration $\log \cdots \log t$ for large enough t. The sequence $B_k = \exp(k/\log_{(m)} k)$ has subexponential growth. Then the corresponding sequence $(\gamma_k)_{k=1}^{\infty}$ satisfies (2), as for large k we have $\gamma_k = \delta_k/\delta_{k-1} < \exp(-2^k B_k) < \exp(-2^k)$ and for the previous k we can take $\gamma_k = 1/32$. By Theorem 7.1, the set $K(\gamma)$ has EP. Let us find a dimension function h that corresponds to this set.

We will search it in the form $h(t) = h_0^{\alpha(t)}(t)$. Let $t = \delta_k$. Then $\log 1/t = 2^{k+1} B_k$, so $k \sim (\log\log 1/t)/\log 2$. On the other hand, $h(t) = 2^{-k} = (2^{k+1}B_k)^{-\alpha(t)}$, which gives $\alpha(t) \sim 1 - (\log 2 \cdot \log_{(m)} k)^{-1} \sim 1 - (\log 2 \cdot \log_{(m+2)} 1/t)^{-1}$. Clearly, $h \succ h_0$.

The next Proposition generalizes Example 3. We restrict our attention to strictly increasing functions h of the form $h = h_0^\alpha$, where α is a monotone function on $[0, t_0]$. As we will be interested in considering dimension functions exceeding h_0 in the next sections, let us suppose that $\alpha(t) \le 1$. Then $h \succ t^\sigma$ for each fixed $\sigma > 0$.

In addition we assume that asymptotically

$$h(t) \le 2\, h(t^2), \tag{25}$$

which is valid for typical dimension functions corresponding to the cases

a) $\alpha(t) = \alpha_0 \in (0, 1]$,
b) $\alpha(t) = \alpha_0 + \varepsilon(t)$ with $\alpha_0 \in [0, 1)$,
c) $\alpha(t) = 1 - \varepsilon(t)$.

Here,

$$\varepsilon(t) \searrow 0 \quad \text{with} \quad \varepsilon(t) \log\log 1/t \nearrow \infty \ \text{ as } t \searrow 0, \tag{26}$$

since for slowly increasing ε we get $h^{\alpha_0 \pm \varepsilon} \approx h_0^{\alpha_0}$.

By (25), for the inverse function h^{-1}, we have $h^{-1}(\tau) \le (h^{-1}(2\tau))^2$ and $h^{-1} \prec \tau^M$ for M given beforehand. From this, $\gamma_k = h^{-1}(2^{-k})/h^{-1}(2^{-k+1})$ defines a sequence satisfying (2). We denote the corresponding set by $K^\alpha(\gamma)$. Our aim is to check *EP* for this set provided regularity of the sequence $B_k = 2^{-k-1} \log(1/h^{-1}(2^{-k}))$. We see at once that B_k increases. In its turn, $\beta_k \searrow 0$ if $\alpha_0 = 1$ in the case (a), $\beta_k \nearrow 1/\alpha_0 - 1$ in (b) and in (a) with $\alpha_0 < 1$. Concerning (c), the monotonicity of β_k requires additional rather technical restrictions on ε. At least for $\varepsilon(t) = \varepsilon_m(t) := (\log_{(m)} 1/t)^{-1}$ we have $\beta_k \searrow 0$. Here, $m \ge 3$, as $h \approx h_0$ for $m \in \{1, 2\}$.

Proposition 8.5. *Let $K^\alpha(\gamma)$ be defined by a function h, as above, with a regular sequence $(B_k)_{k=1}^\infty$. Then $K^\alpha(\gamma)$ has the extension property if and only if*

$$\left(\log \frac{1}{h^{-1}(2^{-k})}\right)^{1/k} \to 2 \ \text{ as } k \to \infty.$$

Proof. Let us find h^{-1} for the case $\alpha(t) = 1 - \varepsilon(t)$. If $h(t) = \tau$ then $[1 - \varepsilon(t)] \log\log 1/t = \log 1/\tau$. Let us define a function δ by the condition $\log\log 1/t = [1 + \delta(\tau)] \log 1/\tau$. Then $[1 - \varepsilon(t)][1 + \delta(\tau)] = 1$, so $\delta(\tau) \searrow 0$ as $\tau \searrow 0$. Then $t = h^{-1}(\tau) = \exp[-(1/\tau)^{1+\delta(\tau)}]$ and $\log(1/h^{-1}(2^{-k})) = 2^{k(1+\delta(2^{-k}))}$. The k-th root of this expression tends to 2. On the other hand, $(B_k)_{k=1}^\infty$ here has subexponential growth as $\beta_k = (\delta(2^{-k}) - 1/k) \log 2 \to 0$. By Theorem 7.1, $K^\alpha(\gamma)$ has *EP*.

Similarly, if $\alpha(t) = \alpha_0 + \varepsilon(t)$ with $0 < \alpha_0 < 1$ then $h^{-1}(\tau) = \exp[-(1/\tau)^{1/\alpha_0 - \delta(\tau)}]$. Here, $(\log(1/h^{-1}(2^{-k})))^{1/k} = 2^{(1/\alpha_0 - \delta(2^{-k}))} \not\to 2$ and $\beta_k \not\to 0$, there is no *EP*. In the case (a), the function δ vanishes.

Lastly, $\alpha_0 = 0$ in (b) gives $h^{-1}(\tau) = \exp[-(1/\tau)^{\Delta(\tau)}]$ with $\Delta(\tau) \nearrow \infty$ as $\tau \searrow 0$. Here, $(\log(1/h^{-1}(2^{-k})))^{1/k} \to \infty$ and $\beta_k \to \infty$. $\square$

9. Extension property and densities of Hausdorff contents

To decide whether a set K has EP, we have to consider a local structure of the most rarefied parts of K. Obviously, such global characteristics as Hausdorff measures or Hausdorff contents cannot be applied in general for this aim. Instead, one can suggest to describe EP in terms of lower densities of M_h or related functions. Given a dimension function h, a compact set K, $x \in K$ and $r > 0$, let $\varphi_{h,K}(x,r) := M_h(K \cap B(x,r))$ and $\varphi_{h,K}(r) := \inf_{x\in K} \varphi_{h,K}(x,r)$, where $B(x,r) = [x-r, x+r]$. One can suppose that K has EP if and only if the corresponding function $\varphi_{h,K}$ is not very small, in a sense, as $r \to 0$. Essentially, this is similar to analysis of the lower density of the Hausdorff content, which can be defined as $\phi_h(K) := \liminf_{r\to 0} \inf_{x\in K} \frac{M_h(K\cap B(x,r))}{M_h(B(x,r))}$. Indeed, $M_h(B(x,r)) = h(2r)$ for h with $h(t) \succ t$ and the expression above is $\liminf_{r\to 0} \frac{\varphi_{h,K}(r)}{h(2r)}$.

In order to distinguish EP by means of ϕ_h, we have to consider large enough dimension functions h. Indeed, if for some h_1 with $h_1 \succ h$ there exists h_1-set K_1 with EP, then h cannot be used for this aim, because $\Lambda_h(K_1) = 0$ implies $M_h(K_1) = 0$ and the corresponding density vanishes contrary to our expectations. Therefore, we can consider only functions exceeding h_0.

We remark that Λ_h-analogs of $\varphi_{h,K}$ or ϕ_h cannot be applied in general for distinguishing EP, since for fat sets ($K = \overline{Int(K)}$) we have $\Lambda_h(K \cap B(x,r)) = \infty$ provided $h(t) \succ t$.

Interestingly, it turns out that the lower density ϕ_h can be used to characterize EP for the family of compact sets considered in [6].

Example 4. Given two sequences $b_k \searrow 0$ (for brevity, we take $b_k = e^{-k}$) and $Q_k \nearrow$ with $Q_k \geq 2$, let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$, where $I_k = [a_k, b_k], |I_k| = b_k^{Q_k}$. In what follows we will consider two cases: $Q_k \leq Q$ with some Q and $Q_k \nearrow \infty$ with $Q_k < \log k$ for large k. By Theorem 4 in [6], K has the extension property in the first case and does not have it for unbounded (Q_k).

In the next lemma we consider concave dimension functions $h = h_0^{\alpha}$ for the cases (a), (b), as above, and for more general

$c')$ $\alpha(t) = \alpha_0 - \varepsilon(t)$ with $\alpha_0 \in (0,1]$.

We suppose now that ε is a monotone differentiable function on $[0, t_0]$ with $0 < \varepsilon(t) < 1 - \alpha_0$ in (b) and $0 < \varepsilon(t) < \alpha_0/2$ in (c'). As before, we assume (26). A direct computation shows that

$$h'(t) < h(t)\, h_0(t)\, \alpha(t)/t \text{ for the cases } (a), (b) \text{ and } \; h'(t) < h(t)\, h_0(t)/t \text{ for } (c'). \tag{27}$$

Lemma 9.1. *Suppose intervals I_k are given as in Example 4 and n is large enough. Then $M_h(\cup_{k=n}^{\infty} I_k) = h(b_n)$. This means that the covering of the set $\cup_{k=n}^{\infty} I_k$ by the interval $[0, b_n]$ is optimal in the sense of definition of M_h.*

Proof. Let us fix a covering of K by open intervals, choose a finite subcovering $\cup_{i=1}^{M} G_i$ and enumerate G_i from left to right. We can suppose that G_1 covers $\cup_{k=N}^{\infty} I_k$ for some $N \geq n$. Indeed, if G_1 covers as well some part of I_{N-1}, then other part of I_{N-1} is covered by G_2. In this case, association of G_1 and G_2 into one interval will give better covering, since $h(b) \leq h(x) + h(b-x)$ for $0 \leq x \leq b$, by concavity of h. For the same reason, we suppose that each G_i covers entire number of I_k. After this we reduce each G_i to the minimal closed interval F_i containing the same intervals I_k. Thus, $F_1 = [0, b_N]$ and $F_2 = [a_{N-1}, b_q]$ with some $n \leq q \leq N-1$. Our aim is to show that

$$h(b_q) < h(b_N) + h(b_q - a_{N-1}), \tag{28}$$

so replacing $F_1 \cup F_2$ with $[0, b_q]$ is preferable. We use the mean value theorem and the decrease of h'. Note that $h(b_k) = k^{-\alpha(b_k)}$.

Consider first the value $q = N - 1$. We will show $h(b_{N-1}) - h(b_N) < h(|I_{N-1}|)$.

In the cases $(a), (b)$, by (27), $LHS < h'(b_N)e^{-N}(e-1) < N^{-1-\alpha(b_N)}\alpha(b_N)(e-1)$. On the other hand, $h(|I_{N-1}|) = [Q_{N-1}(N-1)]^{-\alpha(|I_{N-1}|)}$. Here, $\alpha(|I_{N-1}|) < \alpha(b_N)$, so we reduce the desired inequality to $(Q_{N-1}/N)^{\alpha(b_N)}\alpha(b_N)(e-1) < 1$. It is valid, since for $\alpha_0 > 0$ the first term on the left goes to zero, whereas for $\alpha_0 = 0$ in (b) we have $\alpha(b_N) = \varepsilon(b_N) \to 0$ as $N \to \infty$.

Similarly, in the case (c') the inequality $[Q_{N-1}(N-1)]^{\alpha_0 - \varepsilon(|I_{N-1}|)}(e-1) < N^{1+\alpha_0-\varepsilon(b_N)}$ is valid, as is easy to check.

Suppose now that $q \le N-2$. We write (28) as $h(b_q) - h(b_q - a_{N-1}) < h(b_N)$.

Here, in all cases, by (27), $LHS < h'(b_q - a_{N-1})\, a_{N-1} < h(b_q)h_0(b_q)\frac{a_{N-1}}{b_q - a_{N-1}}$, where the last fraction does not exceed $\frac{b_{N-1}}{b_q - b_{N-1}}$. On the other hand, $h(b_N) \ge N^{-1}$ as $\alpha(b_N) \le 1$. Hence it is enough to show that $N < (e^{N-q-1} - 1)\, q^{1+\alpha(b_q)}$. We neglect $\alpha(b_q)$ and notice that $(e^{N-q-1}-1)\, q \ge (e-1)(N-2)$, which completes the proof of (28).

Continuing in this manner, we see that $h(b_n) \le \sum_{i=1}^{M} h(|F_i|)$. $\square$

Corollary 9.2. *Suppose* $b_{n+1} \le r \le b_n - b_{n+1}$. *Then* $\varphi_{h,K}(r) = h(|I_n|)$.

Proof. Clearly, $\varphi_{h,K}(x,r) = h(|I_n|)$ for each $x \in I_n$. If $x \in K \cap [0, b_{n+1}]$ then $B(x,r)$ covers all intervals I_k with $k \ge n+1$. By Lemma, $\varphi_{h,K}(x,r) = h(b_{n+1}) > h(|I_n|)$. Of course, for $x \in I_k$ with $k < n$ the value $\varphi_{h,K}(x,r)$ also exceed $h(|I_n|)$. $\square$

Remark. The covering of two (or small number of) intervals I_k by one interval is not optimal, since $M_h(I_k \cup I_{k+1}) = h(|I_k|) + h(|I_{k+1}|) < h(b_k - a_{k+1})$.

We proceed to characterize EP for given compact sets in terms of lower densities ϕ_h for $h = h_0^\alpha$, where

$$\alpha(t) = \alpha_0 \in (0,1] \text{ or } \alpha(t) = \alpha_0 \pm \varepsilon_m(t) \tag{29}$$

with $0 < \alpha_0 < 1$ and $\varepsilon_m(t) = (\log_{(m)} 1/t)^{-1}$ for $m > 2$, so (26) is valid.

Proposition 9.3. *Let* K *be from the family of compact sets given in Example 4 and* h *be as above. Then* K *has the extension property if and only if* $\phi_h(K) > 0$.

Proof. Suppose first that $Q_k \le Q$ with some Q, so K has EP. We aim to show $\liminf_{r \to 0} \frac{\varphi_{h,K}(r)}{h(2r)} > 0$. Let $e^{-k-1} \le r < e^{-k}$ for some k. Then, as $\varphi_{h,K}$ increases, $\varphi_{h,K}(r) \ge \varphi_{h,K}(e^{-k-1})$, which is $h(|I_k|) = (k \cdot Q_k)^{-\alpha(|I_k|)}$, by Corollary 9.2. On the other hand, $h(2r) < h(2e^{-k}) = (k - \log 2)^{-\alpha(2e^{-k})}$. Therefore,

$$\varphi_{h,K}(r)/h(2r) > Q_k^{-\alpha(|I_k|)}\; k^{\alpha(2e^{-k}) - \alpha(|I_k|)}\; (1 - \log 2/k)^{-\alpha(2e^{-k})}.$$

The first term on the right converges to $Q^{-\alpha_0}$ as $k \to \infty$. The second and the third terms converge to 1. Hence, $\phi_h(K) \ge Q^{-\alpha_0}$. Besides, this value is achieved in the case $Q_k = Q$ by the sequence $r_k = b_k - b_{k+1}$. Thus, $\phi_h(K) = Q^{-\alpha_0} > 0$.

Similar arguments apply to the case $Q_k \nearrow \infty$, when K does not have EP. Here, $\phi_h(K) \le \lim_k \varphi_{h,K}(r_k)/h(2r_k)$ for r_k as above. By Corollary 9.2, $\varphi_{h,K}(r_k) = h(|I_k|)$. Also, $h(2r_k) > h(e^{-k})$. Hence, $\varphi_{h,K}(r_k)/h(2r_k) < Q_k^{-\alpha_0/2}\, k^{\alpha(e^{-k}) - \alpha(|I_k|)}$, which converges to 0 as k increases. $\square$

Corollary 9.4. *Given* h*, as above, for each* $\sigma > 0$ *there is a compact set with* EP *such that* $0 < \phi_h(K) < \sigma$.

Remark. For this family of sets, the extension property can also be characterized in terms of the Lebesgue linear measure λ. Let $\lambda(r) := \inf_{x\in K} \lambda(K \cap [x-r, x+r])$. Then K has the extension property if and only if $\liminf_{r\to 0} \lambda(r)\cdot r^{-Q} > 0$ for some Q.

Nevertheless, at least for dimension functions $h = h_0^\alpha$ with α as in (29), there is no general characterization of EP in terms of lower densities ϕ_h. In view of Example 3 and the discussion in the beginning of the section, the value $\alpha_0 = 1$ can be omitted from consideration.

We now treat regular sets $K(\gamma)$ with $\delta_k = \exp(-b^k)$. Here, $B_k = 2^{-1}(b/2)^k$. By Theorem 7.1, $K(\gamma)$ has EP if $b = 2$ and does not have it for $b > 2$.

Lemma 9.5. *For each constants $C \geq 1$ and h, as above, there is $b > 2$ such that $h(C\delta_k) < 2\,h(\delta_{k+1})$ for large enough k. This inequality is also valid for $b = 2$.*

Proof. In all cases we have $h(\delta_k) = b^{-k\cdot\alpha(\delta_k)}$ and the desired inequality has the form

$$b^{(k+1)\alpha(\delta_{k+1})} < 2\,(b^k - \log C)^{\alpha(C\delta_k)}. \tag{30}$$

Suppose $\alpha \equiv \alpha_0$. Then (30) is valid as $b^{\alpha_0} < 2\,(1 - b^{-k}\log C)$ for large k and $b = 2+\sigma$ with small enough σ. All the more, it is valid for $b = 2$.

The same reasoning applies to the case $\alpha = \alpha_0 + \varepsilon(t)$ with $\varepsilon \nearrow$ as $\varepsilon(\delta_{k+1}) < \varepsilon(C\delta_k)$.

In the last case $\alpha(t) = \alpha_0 - \varepsilon_m(t)$ we use the following simple inequality

$$\log_{(m)}(Cx) - \log_{(m)}(x) < \log C \cdot [\log x\, \log_{(2)}(x) \cdots \log_{(m-1)}(x)]^{-1},$$

which is valid for all x from the domain of definition of $\log_{(m)}$. From this we have $k\cdot[\varepsilon(C\delta_k) - \varepsilon(\delta_{k+1})] \to 0$ as $k \to \infty$ and (30) can be treated as in the first case. □

Corollary 9.6. *Let k be large enough. Then the covering of each basic interval $I_{j,k}$ of $K(\gamma)$ by one interval is better (in the sense of definition of M_h) than covering by two adjacent subintervals.*

Indeed, by (3), $h(l_{j,k}) < h(C_0\delta_k) < 2\,h(\delta_{k+1}) < h(l_{2j-1,k+1}) + h(l_{2j,k+1})$.

Remark. It is essential that coverings of a whole basic interval are considered. For example, for the set $I_{1,k} \cup I_{3,k+1}$ we have $h(l_{1,k}) + h(l_{3,k+1}) < h(b_{3,k+1})$, which corresponds to the covering of the set by one interval.

Proposition 9.7. *Let $h = h_0^\alpha$ with α as in (29) and $K(\gamma)$ be defined by $\delta_k = \exp(-b^k)$ with $b \geq 2$. Then $\phi_h(K(\gamma)) = b^{-\alpha_0}$.*

Proof. For brevity, we denote here $K(\gamma)$ by K. Fix $x \in K$. Let $x \in I_{j,k} \subset I_{i,k-1}$ and $C_0\,\delta_k \leq r \leq 7/8\cdot\delta_{k-1}$. Then, by (3), $l_{j,k} \leq r < h_{i,k-1}$ and $K \cap [x-r, x+r] = K \cap I_{j,k}$. Arguing as in Lemma 9.1, by Lemma 9.5, we get $\varphi_{h,K}(x,r) = h(l_{j,k})$. Therefore, by monotonicity, $h(\delta_k) < \varphi_{h,K}(x,r) < h(C_0\delta_k)$ for each $x \in K$.

We proceed to estimate $\phi_h(K)$ from both sides. Suppose that $C_0\,\delta_k \leq r \leq C_0\cdot\delta_{k-1}$ for some k. Then $h(\delta_k) < \varphi_{h,K}(r) < h(C_0\delta_{k-1})$ and

$$\frac{h(\delta_k)}{h(2C_0\delta_{k-1})} < \frac{\varphi_{h,K}(r)}{h(2r)} < \frac{h(C_0\delta_{k-1})}{h(2C_0\delta_k)}.$$

Here, $\delta_k = \delta_{k-1}^b$. Analysis similar to that in the proof of Lemma 9.5 shows that the first fraction above has the limit $b^{-\alpha_0}$, whereas the last fraction tends to b^{α_0} as $k \to \infty$. Moreover, the value $b^{-\alpha_0}$ can be achieved as $\lim_k \varphi_{h,K}(r_k)/h(2r_k)$ for $r_k = 7/8\cdot\delta_{k-1}$. □

Comparison of Propositions 9.3 and 9.7 shows that, for given dimension functions, lower densities of Hausdorff contents cannot be used in general to characterize the extension property. Indeed, let us take $K(\gamma)$, as above, with $b > 2$ and K, as in Example 4, with $Q > b$. Then $\phi_h(K) < \phi_h(K(\gamma))$ in spite of the fact that K has EP, whereas $K(\gamma)$ does not.

10. Extension property and growth of Markov's factors

Let $\mathcal{P}_n$ denote the set of all holomorphic polynomials of degree at most n. For any infinite compact set $K \subset \mathbb{C}$ we consider the sequence of *Markov's factors*

$$M_n(K) = \inf\{M : |P'|_{0,K} \le M\,|P|_{0,K},\ P \in \mathcal{P}_n\}$$

for $n \in \mathbb{N}$. We see that $M_n(K)$ is the norm of the operator of differentiation in the space $(\mathcal{P}_n, |\cdot|_{0,K})$. We say that a set K is *Markov* if the sequence $(M_n(K))$ is of polynomial growth. This class of sets is of interest to us, since, by W. Pleśniak [17], any Markov set has EP. On the other hand, there exist non-Markov compact sets with EP ([5,2]). We guess that there is some extremal growth rate $(m_n)_{n=1}^{\infty}$ with the property: if, for some compact set K, $M_n(K)/m_n \to \infty$ as $n \to \infty$ then K does not have EP. The next proposition asserts that here, as above, there is a zone of uncertainty, in which growth rate of Markov's factors is not related with EP. In this sense, it is an analog of Proposition 8.4.

Proposition 10.1. *There are two sets K_1 with EP and K_2 without it, such that $M_n(K_1)$ grows essentially faster than $M_n(K_2)$ as $n \to \infty$.*

Proof. By Theorem 6 in [10], $M_{2^k}(K(\gamma)) \sim 2/\delta_k$. By monotonicity, $\delta_k^{-1} < M_n(K(\gamma)) < 4\,\delta_{k+1}^{-1}$ for $2^k \le n < 2^{k+1}$ with large enough k. As in Proposition 8.4, we take K_1 from Example 1, so $\delta_k^{(1)} = \exp(-2^{k+1}B)$ with $B > 1$. Also, we use K_2 from Example 2 with $A_j = 2^{k_j}$. For simplicity, we fix $k_j = j^2$ that satisfies (22). Here, $\delta_k^{(2)} > k^{-2k}\,\varepsilon_1\varepsilon_2\cdots\varepsilon_j$ for $k_j \le k < k_{j+1}$. We aim to show that $M_n(K_2)/M_n(K_1) \to 0$ as $n \to \infty$. Let us fix large n with $2^k \le n < 2^{k+1}$. For this k we fix j with $k_j \le k < k_{j+1}$. Then

$$M_n(K_2)/M_n(K_1) < 4\,\delta_k^{(1)}/\delta_{k+1}^{(2)}. \tag{31}$$

Suppose first that $k \le k_{j+1}-2$. Then RHS of (31) does not exceed $4\exp[-2^{k+1}B+2(k+1)\log(k+1)+A_j]$. The expression in brackets is smaller than $2^{k_j}(1-2B)+k_{j+1}^2$, which is $(j+1)^4-(2B-1)\,2^{j^2}$, so it tends to $-\infty$ as $j \to \infty$.

If $k = k_{j+1} - 1$ then RHS of (31) is smaller than $4\exp[-2^{k_{j+1}}B + 2k_{j+1}\log k_{j+1} + A_{j+1}]$, which goes to 0, since $B > 1$. This completes the proof. □

Existence of a zone of uncertainty (for the extension property) in the scale of growth rate of Markov's factors implicates the problem to find boundaries of this zone.

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